Representations and characters of the Virasoro algebra and N=1 super-Virasoro algebras [1]

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Abstract

We present the list of irreducible (generalized) highest weight modules over the Virasoro algebra and N=1 super-Virasoro algebras obtained as factor-modules of (generalized) Verma modules. We present also the character formulae of all these modules and single out the unitary irreducible ones. Most formulas are valid for the three algebras under consideration, the different cases being distinguished by two parameters.

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1. Representation theory

The *Virasoro algebra* [1] $\widehat{W}$ is a complex Lie algebra with basis $\hat{c}, L_n$, $n \in \mathbb{Z}$ and Lie brackets:

$$[L_m, L_n] = (m-n)\, L_{m+n} + \delta_{m,-n} \tfrac{1}{12} \left(m^3 - m\right) \hat{c} \qquad (1.1a)$$

$$[\hat{c}, L_n] = 0 \qquad (1.1b)$$

The *Neveu-Schwarz superalgebra* [2] $\widehat{S}$ is a complex Lie superalgebra with basis $\hat{c}, L_n, J_\alpha$, $n \in \mathbb{Z}$, $\alpha \in \mathbb{Z} + \frac{1}{2}$, and Lie (super-)brackets:

$$[L_m, L_n] = (m-n)\, L_{m+n} + \delta_{m,-n} \tfrac{1}{8} \left(m^3 - m\right) \hat{c} \qquad (1.2a)$$

$$[J_\alpha, J_\beta]_+ = 2\, L_{\alpha+\beta} + \delta_{\alpha,-\beta} \tfrac{1}{2} \left(\alpha^2 - \tfrac{1}{4}\right) \hat{c} \qquad (1.2b)$$

$$[L_m, J_\alpha] = (\tfrac{1}{2}m - \alpha)\, J_{m+\alpha} \qquad (1.2c)$$

$$[\hat{c}, L_n] = 0\,, \quad [\hat{c}, J_\alpha] = 0 \qquad (1.2d)$$

The *Ramond superalgebra* [3] $\widehat{R}$ is a complex Lie superalgebra with basis $\hat{c}, L_n$, $n \in \mathbb{Z}$, J_α, $\alpha \in \mathbb{Z}$ and Lie brackets given again by (1.2).

[1] This is a slightly extended version of an Encyclopedia entry.

The Neveu-Schwarz and Ramond superalgebras are also called $N=1$ *super-Virasoro algebras*, since they can be viewed as $N=1$ supersymmetry extensions of the Virasoro algebra.

Further, $\widehat{Q}$ will denote $\widehat{W}$, $\widehat{S}$ or $\widehat{R}$ when a statement holds for all three algebras. The elements $\hat{c}, L_n$ are even and J_α are odd. The grading of $\widehat{Q}$ is given by:

$$\deg \hat{c} \ = \ 0, \quad \deg L_n \ = \ n, \quad \deg J_\alpha \ = \ \alpha \tag{1.3}$$

We have the obvious decomposition:

$$\widehat{Q} \ = \ \widehat{Q}_+ \ \oplus \ \widehat{Q}_0 \ \oplus \ \widehat{Q}_- \tag{1.4}$$

where

$$\widehat{Q}_+ \ = \ \text{c.l.s.}\{ \ X \in \widehat{Q} \ | \ \deg X > 0\} \tag{1.5a}$$

$$\widehat{Q}_0 \ = \ \text{c.l.s.}\{ \ X \in \widehat{Q} \ | \ \deg X = 0\} \tag{1.5b}$$

$$\widehat{Q}_- \ = \ \text{c.l.s.}\{ \ X \in \widehat{Q} \ | \ \deg X < 0\} \tag{1.5c}$$

(c.l.s. means 'complex linear span'). Thus, $\widehat{W}_0$ and $\widehat{S}_0$ are spanned by $\hat{c}$ and L_0, while $\widehat{R}_0$ is spanned by $\hat{c}$, L_0 and J_0. Further, we note that $\widehat{W}_\pm$ is generated by $L_{\pm 1}$ and $L_{\pm 2}$, $\widehat{S}_\pm$ is generated by $J_{\pm\frac{1}{2}}$ and $J_{\pm\frac{3}{2}}$, $\widehat{R}_\pm$ is generated by $L_{\pm 1}$ and $J_{\pm 1}$.

Next we note that the Cartan subalgebra $\widehat{C}$ of $\widehat{Q}$ is spanned by $\hat{c}$ and L_0. We note that $\widehat{C} = \widehat{Q}_0$ for $\widehat{W}$ and $\widehat{S}$. In the case of $\widehat{R}$ the generator J_0 is not in $\widehat{C}$ since it is odd and can not diagonalize $\widehat{R}_\pm$ - having an anti-bracket with the odd generators. Because of this peculiarity we shall use generalized highest weight modules, which will be relevant only in the $\widehat{R}$ case, since in the other two cases they will coincide with ordinary highest weight modules.

A *generalized highest weight module* over $\widehat{Q}$ is characterized by its highest weight $\Lambda \in \widehat{C}^*$ and *generalized highest weight vector* $\tilde{v}_0$, which is a finite-dimensional vector space, so that:

$$\begin{aligned} X\, v \ &= 0\ , \quad X \in \widehat{Q}_+\, , \ v \in \tilde{v}_0 \\ X\, v \ &= \Lambda(X)\, v\ , \quad X \in \widehat{C}, \ v \in \tilde{v}_0 \end{aligned} \tag{1.6}$$

We define the generalized highest weight vector $\tilde{v}_0$ for our three algebras by:

$$\tilde{v}_0 \ = c.l.s. \{ \, v_0 \, \} \, , \quad \text{for} \ \ \widehat{W}, \widehat{S} \tag{1.7a}$$

$$\tilde{v}_0 \ = c.l.s. \{ \, v_0 \, , J_0 \, v_0 \, \} \, , \quad \text{for} \ \ \widehat{R} \tag{1.7b}$$

where v_0 fulfills the conditions (1.6), i.e., it is a usual highest weight vector. We denote:

$$\Lambda(L_0) \;=\; h, \quad \Lambda(\hat{c}) \;=\; c \tag{1.8}$$

Further we shall need also generalized Verma modules over $\widehat{Q}$. For this we introduce bases in the universal enveloping algebras $U(\widehat{Q}_\pm), U(\widehat{Q}_0)$ as follows.

$$J_{\alpha_1} \ldots J_{\alpha_k}\, L_{n_1} \ldots L_{n_\ell}\,, \quad 0 < \alpha_1 < \cdots < \alpha_k\,, \quad 0 < n_1 \leq \cdots \leq n_\ell\,, \quad \text{for } \widehat{Q}_+ \tag{1.9a}$$

$$J_{\alpha_1} \ldots J_{\alpha_k}\, L_{n_1} \ldots L_{n_\ell}\,, \quad \alpha_1 < \cdots < \alpha_k < 0\,, \quad n_1 \leq \cdots \leq n_\ell < 0\,, \quad \text{for } \widehat{Q}_- \tag{1.9b}$$

$$\hat{c}^k\, L_0^\ell\, J_0^\kappa\,, \quad k, \ell \in \mathbb{Z}_+\,, \quad \kappa = 0\,, \quad \text{for } \widehat{W}_0\,, \widehat{S}_0\,, \quad \kappa = 0, 1\,, \quad \text{for } \widehat{R}_0 \tag{1.9c}$$

Naturally, the odd elements appear at most in first degree, since one has (cf. (1.2b)):

$$J_\alpha^2 \;= L_{2\alpha}\,, \qquad \alpha \neq 0 \tag{1.10a}$$

$$J_0^2 \;= L_0 \;-\; \tfrac{1}{16}\,\hat{c}\,, \quad \text{for } \widehat{R}_0 \tag{1.10a}$$

For further reference we say that an element u of $U(\widehat{Q})$ is *homogeneous* if it is an eigenvector of the generator L_0, i.e., if $[L_0, u] \;=\; \lambda_u\, u$. Then we define the *level* of a homogeneous element u of $U(\widehat{Q})$ as $-\lambda_u$. The basis elements in (1.9a,b) are homogeneous of level $-(\alpha_1 + \cdots + \alpha_k + n_1 + \cdots + n_\ell)$, (which is negative, positive, resp., for $U(\widehat{Q}_+)$, $U(\widehat{Q}_-)$, resp.), while those in (1.9c) are homogeneous of level zero.

A generalized Verma module $V^\Lambda = V^{h,c}$ is an induced GHWM with highest weight Λ such that

$$V^\Lambda \;\cong\; U(\widehat{Q}) \otimes_{U(\widehat{B})}\, \tilde{v}_0 \;\cong\; \begin{cases} U(\widehat{Q}_-) \otimes \tilde{v}_0 & \text{for } \widehat{W}_0\,, \widehat{S}_0 \\ U(\widehat{Q}_-) \otimes_{\mathbb{C} J_0}\, \tilde{v}_0 & \text{for } \widehat{R}_0 \end{cases} \tag{1.11}$$

where $\widehat{B} = \widehat{Q}_+ \oplus \widehat{Q}_0$. Below for simplicity we shall write $U(\widehat{Q}_-)\,\tilde{v}_0$ omitting the tensor sign. We denote by $L_\Lambda \;=\; L^{h,c}$ the irreducible factor-module V^Λ / I^Λ, where I^Λ is the maximal proper submodule of V^Λ. Then every irreducible GHWM over $\widehat{Q}$ is isomorphic to some L_Λ.

From now on most formulas will be valid for the three algebras, the different cases being distinguished by two parameters:

$$\mu \;\equiv\; \begin{cases} 0 & \text{for } \widehat{W}, \widehat{S} \\ \frac{1}{2} & \text{for } \widehat{R} \end{cases}\,, \qquad \nu \;\equiv\; \begin{cases} 1 & \text{for } \widehat{W} \\ 2 & \text{for } \widehat{S}, \widehat{R} \end{cases} \tag{1.12}$$

It is known (cf. [4] for $\widehat{W}$, $\widehat{S}$, [5] for $\widehat{R}$) that the generalized Verma module V^{Λ} is reducible iff h and c are related as follows: *either*

$$h \ = \ h_{(m,n)} \ = \ h_0 \ + \ \tfrac{1}{4}(\alpha_+ m + \alpha_- n)^2 + \tfrac{1}{8}\mu \tag{1.13a}$$

where

$$\begin{gathered}
m,n \in \frac{1}{\nu}\, I\!N\,, \quad m-n \in Z\!\!\!Z + \mu \\
h_0 \ = \ \begin{cases} \frac{1}{24}(c-1) & \text{for } \widehat{W} \\ \frac{1}{16}(c-1) & \text{for } \widehat{S}, \widehat{R} \end{cases} \\
\alpha_\pm \ \equiv \ \begin{cases} \frac{1}{\sqrt{24}}\left(\sqrt{1-c} \pm \sqrt{25-c}\right) & \text{for } \widehat{W} \\ \frac{1}{2}\left(\sqrt{1-c} \pm \sqrt{9-c}\right) & \text{for } \widehat{S}, \widehat{R} \end{cases}
\end{gathered} \tag{1.13b}$$

or

$$h = \frac{c}{16} \quad \text{for} \quad \widehat{R} \tag{1.14}$$

For further use we also introduce the notation:

$$c_0 \ \equiv \ \begin{cases} 25 & \text{for } \widehat{W} \\ 9 & \text{for } \widehat{S}, \widehat{R} \end{cases} \tag{1.15}$$

We first comment on the cases from (1.13). We know that the reducible generalized Verma module $V = V^{h,c}$, $h = h_{(m,n)}$, contains a proper submodule isomorphic to the generalized Verma module $V' = V^{h+\nu mn,c}$. In other words there exists a non–trivial embedding map between V' and V. In this situation we shall use the following pictorial representation:

$$V^{h,c} \ \longrightarrow \ V^{h+\nu mn,c}\,, \qquad h = h_{(m,n)} \tag{1.16}$$

These embedding maps are realized by the so called singular vectors. A *singular vector* $v_s \in V$ is such that $v_s \neq \tilde{v}_0$ and v_s has the property of the highest weight vector $\tilde{v}'_0$ of V'. More than this v_s can be expressed, as an element of $U(\widehat{Q}_-)\tilde{v}_0$ by

$$v_s = \mathcal{P}(\widehat{Q}_-)\tilde{v}_0\,, \quad V' \cong U(\widehat{Q}_-)\tilde{v}'_0 \cong U(\widehat{Q}_-)\mathcal{P}(\widehat{Q}_-)\tilde{v}_0\,, \tag{1.17}$$

where $\mathcal{P}(\widehat{Q}_-)$ is a homogeneous polynomial in $U(\widehat{Q}_-)$ of level νmn.

Explicit examples of singular vectors exist for low levels and/or via special constructions, cf. [6], [7], [8], [9], [10], (all for $\widehat{W}$), and also for the cases $h = h_{(m,1)}, h_{(1,n)}$, cf. [11].

Next we comment on (1.14). First we note the following decomposition in the $\widehat{R}$ case which holds for all generalized Verma modules:

$$\begin{aligned} V^\Lambda &= V_0^\Lambda \otimes V_1^\Lambda \\ V_k^\Lambda &= U(\widehat{Q}_-)\ J_0^k\ v_0 \end{aligned} \tag{1.18}$$

Further we consider the action of $\widehat{R}$ on V^Λ. The action of $\widehat{R}_-$ and $\widehat{C}$ preserves each V_k^Λ. Further, we consider separately the cases $h \neq \frac{1}{16}c$ and $h = \frac{1}{16}c$:

1. For $h \neq \frac{1}{16}c$ the action of J_0 is not preserving V_k^Λ. Indeed, it mixes the elements of $\tilde{v}_0$:

$$J_0\,(av_0 + bJ_0v_0) \;=\; b(h - \tfrac{1}{16}c)v_0 + aJ_0v_0 \tag{1.19}$$

Furthermore shifting J_0 to the right until it reaches $\tilde{v}_0$ we have to use:

$$\begin{aligned} J_0\,L_n &= L_n\,J_0 - \tfrac{1}{2}nJ_n\,, \quad n<0 \\ J_0\,J_\alpha &= -\,J_\alpha\,J_0 + 2L_\alpha\,, \quad \alpha<0 \end{aligned} \tag{1.20}$$

Finally, the action of $\widehat{R}_+$ is not preserving V_k^Λ, since shifting its elements to the right one produces also J_0, since:

$$\begin{aligned} L_n\,J_{-n} &= J_{-n}\,L_n + \tfrac{3}{2}nJ_0\,, \quad n>0 \\ J_n\,L_{-n} &= L_{-n}\,J_n + \tfrac{3}{2}nJ_0\,, \quad n>0 \end{aligned} \tag{1.21}$$

2. In the case $h = \frac{1}{16}c$ the subspace V_1^Λ is preserved by the action of $\widehat{R}$. For this it is enough to note that: $J_0(J_0v_0) = (h - \frac{1}{16}c)v_0 = 0$. Thus, the action of J_0 and, consecutively, of $\widehat{R}_+$ on V_1^Λ can not produce elements of V_0^Λ.

Thus, for $h = \frac{1}{16}c$ the generalized Verma module $V = V^{c/16,c}$ over $\widehat{R}$ is reducible. It contains a proper submodule $V_1^{c/16,c}$ which is isomorphic to an ordinary Verma module $\tilde{V} = \tilde{V}^{c/16,c}$ with the same highest weight and highest weight vector $v_0' = J_0v_0$ with the additional property: $J_0\,v_0' = 0$. The factor-module $V^{c/16,c}/V_1^{c/16,c}$ is also isomorphic to $\tilde{V}$ since it has the same highest weight and its highest weight vector $v_0'' = v_0$ also fulfils the property to be annihilated by J_0 - indeed, we have $J_0\,v_0'' = 0$ since $J_0\,v_0 \in V_1^{c/16,c}$. For this reason further in the case $h = \frac{1}{16}c$ we shall consider $\widetilde{V}$ instead of $V^{c/16,c}$ for $\widehat{R}$.

It is also possible that (1.13) and (1.14) hold simultaneously. In this case $\tilde{V}$ is further reducible and everything we said for cases from (1.13) applies also to this case. Furthermore combining (1.13) and (1.14) we have:

$$h \;=\; \tfrac{1}{16}c \;=\; h_{(m,n)} \;=\; \tfrac{1}{16}(c-1) + \frac{1}{4}(m\alpha_+ + na_-)^2 + \tfrac{1}{16} \quad , \quad \text{or,} \tag{1.22a}$$

$$m\alpha_+ + n\alpha_- = 0 \quad , \quad \text{or,} \tag{1.22b}$$

$$-\,\alpha_-/\alpha_+ = \frac{m}{n} = \frac{2m}{2n} = \frac{p}{q} \tag{1.22c}$$

Combining this with the requirement that $m-n \in \mathbb{Z}+\frac{1}{2}$ or $2m-2n \in 2\mathbb{Z}+1$, we see that one of the two numbers $2m, 2n$ must be odd and the other even, and then one of the two numbers p, q must be odd and the other even. Thus, we see that the only possibility in this case is:

$$m \;=\; \tfrac{1}{2}p\,, \quad n \;=\; \tfrac{1}{2}q\,, \quad pq \in 2\mathbb{N}. \tag{1.23}$$

2. Multiplet classification of the reducible (generalized) Verma modules

Here we present the multiplet classification of the reducible (generalized) Verma modules over $\widehat{W}$ [12] (following [13]) and over $\widehat{S}$, $\widehat{R}$ [14]. There are five types in each case which will be denoted (following [14]) $N^0, N^1_+, N^1_-, N^2_+, N^2_-$ They are shown in Table 1. (Note that in [12],[13] the notation was $II, III_\pm\,, III^0_\pm\,, (III^{00}_\pm)$, resp.)

The type N^0 occurs when the ratio α_-/α_+ is not a real rational number.

For all other types the ratio α_-/α_+ is a real rational number and *either* $c \leq 1$ *or* $c \geq c_0$.

For $c<1$ we have type N^1_- and subtypes N^{21}_-, N^{22}_- of type N^2_-. In this case the ratio α_-/α_+ is negative so we set: $\alpha_-/\alpha_+ \;=\; -p/q$, $p,q \in \mathbb{N}$, pXq (the latter means that p,q have no common divisors), and then we have:

$$\begin{aligned} c \;=\; c^-_{p,q} \;&=\tfrac{1}{2}(c_0+1) - \tfrac{1}{4}(c_0-1)\left(\frac{p}{q}+\frac{q}{p}\right) \;=\; 1-(10-4\nu)\frac{(p-q)^2}{pq} \;= \\ &= \;\begin{cases} 13-6\left(\frac{p}{q}+\frac{q}{p}\right) & \text{for } \widehat{W} \\ 5-2\left(\frac{p}{q}+\frac{q}{p}\right) & \text{for } \widehat{S}, \widehat{R} \end{cases}\,, \qquad (c<1) \end{aligned} \tag{2.1}$$

Substituting $c \longrightarrow c^-_{p,q}$ in (1.13a) we get:

$$h \;=\; h^-_{(m,n)} \;=\; \frac{1}{4\nu pq}[\nu^2(pn-qm)^2 - (p-q)^2] + \tfrac{1}{8}\mu \tag{2.2}$$

For $c=1$ we have subtype N^{23}_- of type N^2_-.

For $c>c_0$ we have type N^1_+ and subtypes N^{21}_+, N^{22}_+ of type N^2_+. Here the ratio α_-/α_+ is positive so we set: $\alpha_-/\alpha_+ \;=\; p/q$, $p,q \in \mathbb{N}$, pXq, and then we have:

$$\begin{aligned} c \;=\; c^+_{p,q} \;&=\tfrac{1}{2}(c_0+1) + \tfrac{1}{4}(c_0-1)\left(\frac{p}{q}+\frac{q}{p}\right) \;=\; 1+(10-4\nu)\frac{(p-q)^2}{pq} \;= \\ &= \;\begin{cases} 13+6\left(\frac{p}{q}+\frac{q}{p}\right) & \text{for } \widehat{W} \\ 5+2\left(\frac{p}{q}+\frac{q}{p}\right) & \text{for } \widehat{S}, \widehat{R} \end{cases}\,, \qquad (c>c_0) \end{aligned} \tag{2.3}$$

Substituting $c \to c^+_{p,q}$ in $(1.13a)$ we get:

$$h \;=\; h^+_{(m,n)} \;=\; \frac{1}{4\nu pq}[(p+q)^2 - \nu^2(pn-qm)^2] + \tfrac{1}{8}\mu \;=\; \frac{1}{\nu} + \tfrac{1}{4}\mu - h^-_{(m,n)} \tag{2.4}$$

For $c = c_0$ we have subtype N^{23}_+ of type N^2_+ .

The explicit parametrization of all types and subtypes is given in Table 2 (for $\widehat{W}$ see [12] and Propositions 1.1-1.5, formulae (6),(13),(16) of [13], for $\widehat{S}, \widehat{R}$ see Propositions 1-4, formulae (6),(11),(14) of [14]). Note that in each case the (sub)types N_- with $c \le 1$ have the same parametrization as the corresponding N_+ with $c \to c_0 + 1 - c \ge c_0$. Further, we note that the parametrization of subtypes $N^{21}_\pm$ is obtained from the parametrization of $N^1_\pm$ by formally setting $n = \tilde{q}$ and replacing the condition $p < q$ by $p \ne q$. Next, the parametrization of subtypes $N^{22}_\pm$ is obtained from the parametrization of $N^{21}_\pm$ by formally setting $m = \tilde{p}$ or $m = 0$. Finally, the parametrization of subtypes $N^{23}_\pm$ is obtained from the parametrization of $N^{22}_\pm$ by formally setting $p = q$ (which forces $p = q = 1$).

Further we give the explicit parametrization of the (generalized) Verma modules in the different multiplets. The cases N^0 are simple and all info about then is already present in Tables 1,2.

We start with type N^1_- . For fixed parameters p, q, m, n (cf. Table 2) the (generalized) Verma modules of this type form a multiplet represented by the corresponding commutative diagram of Table 1. The modules of this multiplet are divided in four infinite groups which are given as follows (cf. [13], formula (10), [14], formula (10)):

$$V_{0k} \;=\; V^{h_{0k}\,,c}, \quad k \in \mathbb{Z}_+ \tag{2.5a}$$

$$h_{0k} \;= h_{00} + \nu k(\tilde{p}\tilde{q}k + \tilde{q}m - \tilde{p}n) \;=\; h^-_{(2k\tilde{p}+m,n)} \;=\; h^-_{(\tilde{p}-m,2k\tilde{q}+\tilde{q}-n)} \tag{2.5a$'$}$$

$$V_{1k} \;=\; V^{h_{1k}\,,c}, \quad k \in \mathbb{Z}_+ \tag{2.5b}$$

$$h_{1k} \;= h_{00} + \nu k(\tilde{p}\tilde{q}k - \tilde{q}m + \tilde{p}n) \;=\; h^-_{(m,2k\tilde{q}+n)} \;=\; h^-_{(2k\tilde{p}+\tilde{p}-m,\tilde{q}-n)} \tag{2.5b$'$}$$

$$V'_{0k} \;=\; V^{h'_{0k}\,,c}, \quad k \in \mathbb{Z}_+ \tag{2.5c}$$

$$h'_{0k} \;= h_{00} + \nu(\tilde{q}k + n)(\tilde{p}k + m) \;=\; h^-_{(2k\tilde{p}+\tilde{p}+m,\tilde{q}-n)} \;=\; h^-_{(\tilde{p}-m,2k\tilde{q}+\tilde{q}+n)} \tag{2.5c$'$}$$

$$V'_{1k} \;=\; V^{h'_{1k}\,,c}, \quad k \in \mathbb{Z}_+ \tag{2.5d}$$

$$h'_{1k} \;= h_{00} + \nu(\tilde{q}k + \tilde{q} - n)(\tilde{p}k + \tilde{p} - m) \;=\; h^-_{(m,2k\tilde{q}+2\tilde{q}-n)} \;=\; h^-_{(2k\tilde{p}+2\tilde{p}-m,n)} \tag{2.5d$'$}$$

where $c = c^-_{p,q}$, and we note that $h_{00} = h_{10}$.

The (generalized) Verma modules of type N^1_+ for fixed parameters p, q, m, n also form a multiplet represented by the corresponding commutative diagram of Table 1. The modules

of this multiplet are also divided in four infinite groups which we denote by V^+_{0k} , V'^+_{0k} , V^+_{1k} , V'^+_{1k} , which are given by (2.5) with the changes $\tilde p \to -\tilde p, m \to -m$ (or $\tilde q \to -\tilde q, n \to -n$), $h \to h^+_{(m,n)}$, $c \to c^+_{p,q}$.

We continue with subtype N^{21}_- . For fixed parameters p, q, m (cf. Table 2) the (generalized) Verma modules of this type form a multiplet represented by the corresponding diagram of Table 1. The modules of this multiplet are divided in two infinite groups whose parametrization is obtained from those of type N^1_- by setting $n = \tilde q$:

$$V^0_{0k} = (V_{0k})_{|n=\tilde q} = \left(V'_{0,k-1}\right)_{|n=\tilde q\,,\,k\geq 1} = V^{h^0_{0k}\,,\,c}, \quad k \in \mathbb{Z}_+ \tag{2.6a}$$

$$h^0_{0k} = h^0_{00} + \nu k\tilde q(\tilde p k + m - \tilde p) = h^-_{(\tilde p - m, 2k\tilde q)} \tag{2.6a$'$}$$

$$V^0_{1k} = (V_{1k})_{|n=\tilde q} = (V'_{1k})_{|n=\tilde q} = V^{h^0_{1k}\,,\,c}, \quad k \in \mathbb{Z}_+ \tag{2.6b}$$

$$h^0_{1k} = h^0_{10} + \nu k\tilde q(\tilde p k - m + \tilde p) = h^-_{(m,(2k+1)\tilde q)} \tag{2.6b$'$}$$

where $h^0_{0k} = h^0_{1k} = h^-_{(m,\tilde q)} = \frac{1}{4\nu pq}[\nu^2 q^2(\tilde p - m)^2 - (p-q)^2] + \frac{1}{8}\mu$ and $c = c^-_{p,q}$. Note that $V^0_{00} = V^0_{10}$.

The multiplet of subtype N^-_{21} describes also the situation for $\widehat R$ when both (1.13) and (1.14) hold, and then (1.23) holds. We shall denote the corresponding multiplet by R^-_{21} to stress that it happens only for $\widehat R$. The modules of this multiplet are also divided in two infinite groups whose parametrization is obtained from those of type N^1_- by using (1.23). Thus, we get:

$$\widetilde V^0_{0k} = \left(\widetilde V_{0k}\right)_{|_{(m,n)=(\frac p2,\frac q2)}} = \left(\widetilde V_{1k}\right)_{|_{(m,n)=(\frac p2,\frac q2)}} = \widetilde V^{\tilde h^0_{0k}\,,\,c}, \quad k \in \mathbb{Z}_+ \tag{2.7a}$$

$$\tilde h^0_{0k} = \tilde h^0_{00} + 2pqk^2 = h^-_{\left((4k+1)\frac p2,\frac q2\right)} = h^-_{\left(\frac p2,(4k+1)\frac q2\right)} \tag{2.7a$'$}$$

$$\widetilde V^0_{1k} = \left(\widetilde V'_{0k}\right)_{|_{(m,n)=(\frac p2,\frac q2)}} = \left(\widetilde V'_{1k}\right)_{|_{(m,n)=(\frac p2,\frac q2)}} = \widetilde V^{\tilde h^0_{1k}\,,\,c}, \quad k \in \mathbb{Z}_+ \tag{2.7b}$$

$$\tilde h^0_{1k} = \tilde h^0_{10} + 2pq(k+\tfrac12)^2 = h^-_{\left((4k+3)\frac p2,\frac q2\right)} = h^-_{\left(\frac p2,(4k+3)\frac q2\right)} \tag{2.7b$'$}$$

where $\tilde h^0_{00} = h^-_{(\frac p2,\frac q2)} = -\frac{1}{8pq}(p-q)^2 + \frac{1}{16} = \frac{c}{16} = \frac{1}{16}c^-_{p,q}$. We remind that the modules here are ordinary Verma modules with highest weight vector $J_0\,v_0$. Note that the tildes are omitted in Table 1 since we use the same diagram as for N^-_{21} .

The (generalized) Verma modules of subtype N^{21}_+ for fixed parameters p, q, m also form a multiplet represented by the corresponding diagram of Table 1. The modules of this multiplet are also divided in two groups which we denote by V^{+0}_{0k} , V^{+0}_{1k} , which are given by

(2.6) with the changes $\tilde{p} \to -\tilde{p}, m \to -m,\ h \to h^+_{(m,\tilde{q})} = \frac{1}{4\nu pq}[(p+q)^2 - \nu^2 q^2(\tilde{p}-m)^2] + \frac{1}{8}\mu$, $c \to c^+_{p,q}$.

The multiplet N^+_{21} describes also the situation for $\widehat{R}$ when both (1.13) and (1.14) hold, and then (1.23) holds. The corresponding multiplet is denoted by R^+_{21} to stress that it happens only for $\widehat{R}$. The modules of this multiplet are also divided in two infinite groups whose parametrization is obtained from those of type N^1_+ by using (1.23). Thus, we get:

$$\begin{aligned} \widetilde{V}^{+0}_{0k} &= \left(\widetilde{V}^+_{0k}\right)_{|_{(m,n)=(\frac{p}{2},\frac{q}{2})}} = \left(\widetilde{V}^+_{1k}\right)_{|_{(m,n)=(\frac{p}{2},\frac{q}{2})}} = \widetilde{V}^{\tilde{h}^{+0}_{0k},c} && (2.8a) \\ \tilde{h}^{+0}_{0k} &= \tilde{h}^{+0}_{00} - 2pqk^2 && (2.8a') \\ \widetilde{V}^{+0}_{1k} &= \left(\widetilde{V}'^+_{0k}\right)_{|_{(m,n)=(\frac{p}{2},\frac{q}{2})}} = \left(\widetilde{V}'^+_{1k}\right)_{|_{(m,n)=(\frac{p}{2},\frac{q}{2})}} = \widetilde{V}^{\tilde{h}^{+0}_{1k},c} && (2.8b) \\ \tilde{h}^{+0}_{1k} &= \tilde{h}^{+0}_{10} - 2pq(k+\tfrac{1}{2})^2 && (2.8b') \end{aligned}$$

where $k \in \mathbb{Z}_+$, $\tilde{h}^{+0}_{00} = h^+_{(\frac{p}{2},\frac{q}{2})} = \frac{1}{8pq}(p+q)^2 + \frac{1}{16} = \frac{c}{16} = \frac{1}{16}c^+_{p,q}$. We remind that the modules here are ordinary Verma modules with highest weight vector $J_0\, v_0$. Note that the tildes are omitted in Table 1 since we use the same diagram as for N^+_{21} .

We continue with subtype N^{22}_- . For fixed parameters p, q, ϵ (cf. Table 2) the (generalized) Verma modules of this subtype form a multiplet represented by the corresponding diagram of Table 1. The modules of the multiplet are an infinite set whose parametrization is obtained from those of subtype N^{21}_- by setting $m = \tilde{p}$ when $\epsilon = 0$:

$$\begin{aligned} V^0_k &= (V^0_{0k})_{|m=\tilde{p}} = (V^0_{1k})_{|m=\tilde{p}} = V^{h^0_k,c} && (2.9a) \\ h^0_k &= h^0_0 + \nu\tilde{q}\tilde{p}k^2 = h^-_{(\tilde{p},(2k+1)\tilde{q})} \end{aligned}$$

where $k \in \mathbb{Z}_+$, $h^0_0 = h^-_{(\tilde{p},\tilde{q})} = -\frac{1}{4\nu pq}(p-q)^2 + \frac{1}{8}\mu$ and $c = c^-_{p,q}$, or setting $m = 0$ when $\epsilon = 1$:

$$\begin{aligned} V^1_k &= (V^0_{1k})_{|m=0} = (V^0_{0,k+1})_{|m=0} = V^{h^1_k,c} && (2.9b) \\ h^1_k &= h^1_0 + \nu\tilde{p}\tilde{q}k(k+1) = h^-_{(\tilde{p},2(k+1)\tilde{q})} \end{aligned}$$

where $k \in \mathbb{Z}_+$, $h^1_0 = h^-_{(0,\tilde{q})} = \frac{1}{4\nu pq}[\nu^2 p^2 \tilde{q}^2 - (p-q)^2] + \frac{1}{8}\mu$ and $c = c^-_{p,q}$. Note that the case $(2.9a, a')$ it is necessary that $\tilde{p} - \tilde{q} \in \mathbb{Z} + \mu$, which excludes the $\widehat{R}$ case since $\tilde{p} - \tilde{q} \in \mathbb{Z}$ in all cases. In the case $(2.9b, b')$ it is necessary that $\tilde{p} \in \mathbb{Z} + \mu$, which excludes the $\widehat{R}$ case if $pq \in 2\mathbb{N}$ and excludes the $\widehat{S}$ case if $pq \in 2\mathbb{N} + 1$.

The (generalized) Verma modules of subtype N_+^{22} for fixed parameters p, q, ϵ also form a multiplet represented by the corresponding diagram of Table 1. The modules of this multiplet also form an infinite set we denote by $V_k^{+\epsilon}$, whose parametrization is obtained from (2.9,)(2.9,) resp., for $\epsilon = 0, 1$, resp., with the changes $h \to h^+_{(\tilde{p},\tilde{q})} = \frac{1}{4\nu pq}(p+q)^2 + \frac{1}{8}\mu$, $h \to h^+_{(0,\tilde{q})} = \frac{1}{4\nu pq}[(p+q)^2 - \nu^2 q^2 \tilde{p}^2] + \frac{1}{8}\mu$, resp., and $\tilde{p} \to -\tilde{p}$, $c \to c^+_{p,q}$.

We continue with subtype N_-^{23}. For fixed parameter $\epsilon = 0, 1$ (cf. Table 2) the (generalized) Verma modules of this subtype form a multiplet represented by the corresponding diagram of Table 1. The (generalized) Verma modules of this subtype form an infinite set whose parametrization is obtained from those of subtype N_-^{22} by setting $p = q = 1$; then $\tilde{p} = \tilde{q} = \frac{1}{\nu}$; $c = c^-_{1,1} = 1$. We have for $\epsilon = 0$:

$$V_k^0 \ = \ V^{h + \frac{1}{\nu}k^2, 1} \tag{2.10}$$

where $k \in \mathbb{Z}_+$, $h = h^-_{(\frac{1}{2},\frac{1}{2})} = \frac{1}{8}\mu = 0$. The last equality is because this case is possible only for $\widehat{W}, \widehat{S}$. For $\epsilon = 1$ we have:

$$V_k^1 \ = \ V^{h + \frac{1}{\nu}k(k+1), 1} \tag{2.11}$$

where $k \in \mathbb{Z}_+$, $h = h^-_{(0,\frac{1}{2})} = \frac{1}{4\nu} + \frac{1}{8}\mu$. This case is possible only for $\widehat{W}, \widehat{R}$.

Finally, we consider subtype N_+^{23}. For fixed parameter $\epsilon = 0, 1$ (cf. Table 2) the (generalized) Verma modules of this subtype form a multiplet represented by the corresponding diagram of Table 1. The (generalized) Verma modules of this subtype form an infinite set whose parametrization is obtained from those of subtype N_+^{22} by setting $p = q = 1$; then $\tilde{p} = \tilde{q} = \frac{1}{\nu}$; $c = c^+_{1,1} = c_0$. We have for $\epsilon = 0$:

$$V_k^{+0} \ = \ V^{h - \frac{1}{\nu}k^2, 1} \tag{2.12}$$

where $k \in \mathbb{Z}_+$, $h = h^+_{(\frac{1}{2},\frac{1}{2})} = \frac{1}{\nu} + \frac{1}{8}\mu = \frac{1}{\nu}$. The last equality is because this case is possible only for $\widehat{W}, \widehat{S}$. For $\epsilon = 1$ we have:

$$V_k^1 \ = \ V^{h - \frac{1}{\nu}k(k+1), 1} \tag{2.13}$$

where $k \in \mathbb{Z}_+$, $h = h^+_{(0,\frac{1}{2})} = \frac{3}{4\nu} + \frac{1}{8}\mu$. This case is possible only for $\widehat{W}, \widehat{R}$.

3. Characters of (generalized) highest weight modules

We recall the weight space decomposition of $V^{h,c}$

$$V^{h,c} \ = \ \mathop{\oplus}_j V_j^{h,c}, \quad j \in \mathbb{Z}_+ \ \text{ for } \ \widehat{W}, \widehat{R}, \quad j \in \tfrac{1}{2}\mathbb{Z}_+ \quad \text{for } \widehat{S} \ , \tag{3.1}$$

where $V_j^{h,c}$ are eigenspaces of L_0

$$V_j^{h,c} \ = \ \{v \in V^{h,c} | L_0 v = (h+j)v\} \cong U(\widehat{Q}_-)_j \tilde{v}_0, \tag{3.2}$$

where the last equality follows from

$$U(\widehat{Q}_-) = \mathop{\oplus}_j U(\widehat{Q}_-)_j \tag{3.3}$$

with the range of j as in (16). For $\widehat{R}$ we have also

$$\begin{aligned} &V_j^{h,c} = V_{j,1}^{h,c} \oplus V_{j,2}^{h,c} \ , \\ &V_{j,1}^{h,c} = U(\widehat{Q}_-)_j \, J_0 v_0, \quad V_{j,2}^{h,c} = U(\widehat{Q}_-)_j \, v_0 \ . \end{aligned} \tag{3.4}$$

The character of $V^{h,c}$ is defined (cf. [4],[5]) as

$$ch \ V^{h,c}(t) \ = \ \sum_j (\dim V_j^{h,c}) \, t^{h+j} = t^h \sum_j p(j) \, t^j = t^h \psi(t) \ , \tag{3.5}$$

where $p(j)$ is the partition function ($p(j) = \#$ of ways j can be represented as the sum of positive integers (and half–integers for $\widehat{S}$); $p(0) \equiv 1$), while $\psi(t)$ is given by [4],[5]:

$$\psi(t) = \begin{cases} \prod_{k \in \mathbb{N}} (1-t^k)^{-1} & \text{for } \widehat{W} \\ \prod_{k \in \mathbb{N}} (1+t^{k-1/2})/(1-t^k) & \text{for } \widehat{S} \\ \prod_{k \in \mathbb{N}} \ 2(1+t^k)/(1-t^k) & \text{for } \widehat{R} \end{cases} \tag{3.6}$$

The factor of 2 for $\widehat{R}$ appears because of the relation

$$\dim V_j^{h,c} = (1+2\mu) \dim U(\widehat{Q}_-)_j \ . \tag{3.7}$$

For $\widehat{R}$ and $h = c/16$ we have for $\tilde{V} = \tilde{V}^{c/16,c} \equiv U(\tilde{R}_-) \, J_0 v_0$

$$ch \ \tilde{V}(t) = t^{c/16} \prod_{k \in \mathbb{N}} (1+t^k)/(1-t^k) \ . \tag{3.8}$$

We present now the character formulae for the irreducible GHWM L_Λ in the cases when $L_\Lambda \neq V^\Lambda$ (for $\widehat{W}$ cf. [15], [16], [17], (for partial results see [4], [18], [19], [20], [21]); for $\widehat{S}, \widehat{R}$ cf. [17], (for partial results see [4], [20], [22])).

In the N^0 case the embedded $V^{h+\nu mn,c}$ is irreducible while for $L^{h,c}$ using the results of [16] one can obtain

$$ch\, L^{h,c} \;=\; ch\, V^{h,c} - ch\, V^{h+\nu mn,c} \;=\; (1 - t^{\nu mn})\; ch\, V^{h,c}\,. \tag{3.9}$$

In the N^1_- case let us denote by $L_{0k}, L_{1k}, L'_{0k}, L'_{1k}$ the irreducible factor–modules of $V_{\ell k}, V'_{\ell k}$, resp. Then we have ([15], [16], [17], formula (22)):

$$ch\, L_{\ell k} \;=\; ch\; V_{\ell k} + \sum_{j>k}(ch\; V_{0j} + ch\; V_{1j}) - \sum_{j\geq k}(ch\; V'_{0j} + ch\; V'_{1j}), \tag{3.10a}$$

$$ch\, L'_{\ell k} \;=\; ch\, V'_{\ell k} + \sum_{j>k}(ch\; V'_{0j} + ch\; V'_{1j} - ch\; V_{0j} - ch\; V_{1j}), \tag{3.10b}$$

where in both formulae $\ell = 0,1,\;\; k \in \mathbb{Z}_+$.

In the N^1_+ case we denote by $L^+_{\ell k}, L^{'+}_{\ell k}$, $(\ell = 0,1)$, the irreducible factor–modules of $V^+_{\ell k}, V'^+_{\ell k}$ resp. Then we have ([15], [16], [17], formula (35)):

$$ch\, L^+_{\ell k} \;= ch\, V^+_{\ell k} + \sum_{j=1}^{k-1}(ch\, V_{0j} + ch\, V_{1j}) + ch\, L^+_{00} - \sum_{j=0}^{k-1}(ch\, V'_{0j} + ch\, V'_{1j}), \quad k>0 \tag{3.11a}$$

$$ch\, L'^+_{\ell k} \;= ch\, V'^+_{\ell k} + \sum_{j=0}^{k-1}(ch\, V'_{0j} + ch\, V'_{1j}) - ch\, L^+_{00} - \sum_{j=1}^{k}(ch\, V_{0j} + ch\, V_{1j}), \quad k\geq 0 \tag{3.11b}$$

where in both formulae $\ell = 0,1$; ($ch\, L^+_{00} = ch\, L^+_{10} = ch\, V^+_{00} = ch\, V^+_{10}$, since $V^+_{00} = V^+_{10}$ is irreducible).

In the $N^{21}_\pm$ cases let us denote by $L^0_{\ell k}$, $L^{+0}_{\ell k}$ the irreducible factor module of $V^0_{\ell k}$, $V^{+0}_{\ell k}$, resp. Then we have ([15], [16], [17], formulae (28), (29)):

$$\begin{aligned} ch\, L^0_{0k} &= ch\, V^0_{0k} - ch\, V^0_{1k}, \quad k>0, \\ ch\, L^0_{1k} &= ch\, V^0_{1k} - ch\, V^0_{0,k+1}, \quad k\geq 0, \end{aligned} \tag{3.12}$$

and ([15], [16], [17], formulae (42), (43)):

$$\begin{aligned} ch\, L^{+0}_{0k} &= ch\, V^{+0}_{0k} - ch\, V^+_{1,k-1}, \quad k>0, \\ ch\, L^{+0}_{1k} &= ch\, V^{+0}_{1k} - ch\, V^+_{0k}, \quad k>0, \end{aligned} \tag{3.13}$$

($ch\, L_{00}^{+0} = ch\, L_{10}^{+0} = ch\, V_{00}^{+0} = ch\, V_{10}^{+0}$, since $V_{00}^{+0} = V_{10}^{+0}$ is irreducible.) These formulae are valid also for the $R_\pm^{21}$ cases but all generalized Verma modules $V_{\ell k}^0$, $V_{\ell k}^{+0}$ should be replaced with the ordinary Verma modules $\widetilde{V}_{\ell k}^0$, $\widetilde{V}_{\ell k}^{+0}$, (with highest weight vector $J_0\, v_0$ as discussed above).

In the $N_\pm^{22}$, $N_\pm^{23}$, cases let us denote by L_k^ϵ, $L_k^{+\epsilon}$ the irreducible factor module of V_k^ϵ, $V_k^{+\epsilon}$, resp. Then we have ([15], [16], [17], formula (31)):

$$ch\, L_k^\epsilon \;=\; ch\, V_k^\epsilon - ch\, V_{k+1}^\epsilon, \quad k \geq 0, \tag{3.14}$$

and ([15], [16], [17], formulae (44), (45)):

$$ch\, L_k^{+\epsilon} \;=\; ch\, V_k^{+\epsilon} - ch\, V_{k-1}^{+\epsilon}, \quad k > 0. \tag{3.15}$$

($ch\, L_0^{+\epsilon} \;=\; ch\, V_0^{+\epsilon}$ since $V_0^{+\epsilon}$ is irreducible.)

4. Unitarity

The irreducible factor-modules $L_\Lambda \;=\; L^{h,c} \;=\; V^\Lambda / I^\Lambda$ are unitarizable [5],[22],[23] if *either* $h \geq 0$, $c \geq 1$, *or* when

$$h \;=\; \frac{1}{4\nu p(p+\nu)}[\nu^2(pn - m(p+\nu))^2 - \nu^2] + \tfrac{1}{8}\mu\,, \quad c \;=\; 1 - \tfrac{2(2+\nu)}{p(p+\nu)}\,, \quad p = 2,3,\ldots \tag{4.1}$$

or when

$$h \;=\; h^-_{(\frac{p}{2},\frac{p+1}{2})} \;=\; \tfrac{1}{16} c^-_{p,p+1} \;=\; \tfrac{1}{16}\left(1 - \tfrac{2}{p(p+1)}\right)\,, \quad p = 1,2,\ldots \qquad \text{for} \quad \widehat{R}\,. \tag{4.2}$$

The cases (4.1) are the modules $L_{00} \equiv L_{10}$ (parametrized by (m,n) as in Table 2) which form the $c<1$ series of unitarizable HWM over $\widehat{W}$, [5], [23], $\widehat{S}$, $\widehat{R}$ [5],[22] inside the so-called Kac table [6]. The case (4.2) is the module $L_{00}^0 \equiv L_{10}^0$ on the border of the Kac table.

In the cases (4.1) for $p=2$ one gets the trivial case $c=0$. The cases $p = 3,4,5,6$ for the Virasoro algebra ($\nu = 1$), i.e., $c \;=\; 1/2,\ 7/10,\ 4/5,\ 5/6$, resp., correspond to the Ising model, tri-critical Ising model, 3-state Potts model, tri-critical 3-state Potts model, resp., [5]. Note that there is only one value of the central charge which is common for the Virasoro and super-Virasoro algebras. This is the case $c = 7/10$ for Virasoro ($p=4$) and $c = 7/15$ for the super-Virasoro algebras ($p=3$), taking into account that $c_{\widehat{W}} \;=\; \frac{3}{2}\, c_{\widehat{S},\widehat{R}}$ (compare (1.1a) and (1.2a)).

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Table 1

Types of multiplets (embedding diagrams) of reducible Verma modules over the Virasoro and $N=1$ super–Virasoro algebras (arrows point *to* the embedded modules)

N^0

$$V^{h,c} \longrightarrow V^{h+\nu mn,c}$$

N^1_-

$$\begin{tikzcd}
 & V'_{00} \arrow[r] \arrow[ddr] & \cdots \arrow[r] \arrow[ddr] & V_{0k} \arrow[r] \arrow[ddr] & V'_{0k} \arrow[r] \arrow[ddr] & \cdots \\
V_{00} = V_{10} \arrow[ur] \arrow[dr] & & & & & \\
 & V'_{10} \arrow[r] \arrow[uur] & \cdots \arrow[r] \arrow[uur] & V_{1k} \arrow[r] \arrow[uur] & V'_{1k} \arrow[r] \arrow[uur] & \cdots
\end{tikzcd}$$

N^1_+

$$\begin{tikzcd}
 & V'^{+}_{00} \arrow[dl] & \cdots \arrow[l] \arrow[ddl] & V^{+}_{0k} \arrow[l] \arrow[ddl] & V'^{+}_{0k} \arrow[l] \arrow[ddl] & \cdots \arrow[l] \arrow[ddl] \\
V^{+}_{00} = V^{+}_{10} & & & & & \\
 & V'^{+}_{10} \arrow[ul] & \cdots \arrow[l] \arrow[uul] & V^{+}_{1k} \arrow[l] \arrow[uul] & V'^{+}_{1k} \arrow[l] \arrow[uul] & \cdots \arrow[l] \arrow[uul]
\end{tikzcd}$$

$N^{21}_-\,, R^{21}_-$

$$V^0_{00} = V^0_{10} \longrightarrow V^0_{01} \longrightarrow V^0_{11} \longrightarrow \cdots \longrightarrow V^0_{0k} \longrightarrow V^0_{1k} \longrightarrow \cdots$$

$N^{21}_+\,, R^{21}_+$

$$V^{+0}_{00} = V^{+0}_{10} \longleftarrow V^{+0}_{01} \longleftarrow V^{+0}_{11} \longleftarrow \cdots \longleftarrow V^{+0}_{0k} \longleftarrow V^{+0}_{1k} \longleftarrow \cdots$$

$N^{22}_-\,, N^{23}_-$

$$V^{\epsilon}_0 \longrightarrow V^{\epsilon}_1 \longrightarrow \cdots \longrightarrow V^{\epsilon}_k \longrightarrow \cdots$$

$N^{22}_+\,, N^{23}_+$

$$V^{+\epsilon}_0 \longleftarrow V^{+\epsilon}_1 \longleftarrow \cdots \longleftarrow V^{+\epsilon}_k \longleftarrow \cdots$$

Table 2

Parametrization of the types of multiplets of (generalized) Verma modules over the Virasoro [12],[13] and $N=1$ super-Virasoro algebras [14]

type	parameters	range and constraints on parameters	remarks
N^0 (II)	c,m,n	$c\in\mathbb{C},\quad \alpha_-/\alpha_+\notin\mathbb{Q}$, $m,n\in\frac{1}{\nu}\mathbb{N},\quad m-n\in\mathbb{Z}+\mu$	
$N^1_\pm$ ($\mathrm{III}_\pm$)	p,q,m,n	$p,q\in\mathbb{N},\ \ pXq,\ \ p<q$, $m,n\in\frac{1}{\nu}\mathbb{N},\ m-n\in\mathbb{Z}+\mu$, $pn>qm,\ n<\tilde{q}$, $\tilde{q}\equiv\begin{cases}\frac{q}{2} & \text{if } pq\in 2\mathbb{N}+1 \text{ for } \widehat{S},\widehat{R}\\ q & \text{otherwise}\end{cases}$	$\alpha_-/\alpha_+=\pm p/q\,$; $c=c^\pm_{p,q}\,$;
$N^{21}_\pm$ ($\mathrm{III}^0_\pm$)	p,q,m	$p,q\in\mathbb{N},\ pXq,\ p\neq q,\quad m\in\frac{1}{\nu}\mathbb{N}$ $m<\tilde{p}\neq\tilde{q},\quad m-\tilde{q}\in\mathbb{Z}+\mu$	$\alpha_-/\alpha_+=\pm p/q$ or $\pm q/p$; $c=c^\pm_{pq}\,$;
$R^{21}_\pm$	p,q	$p,q\in\mathbb{N},\ pXq,\ p\neq q,\ pq\in 2\mathbb{N}$ $h=h^\pm_{(\frac{p}{2},\frac{q}{2})}=\frac{1}{16}c^\pm_{p,q}$, only for $\widehat{R}$	$\alpha_-/\alpha_+=\pm p/q,\ c=c^\pm_{pq}\,$; the only possibility for $\tilde{V}^{\frac{c}{16},c}$
$N^{22}_\pm$ ($\mathrm{III}^0_\pm$)	p,q,ϵ	$p,q\in\mathbb{N},\ pXq,\ p<q$, $\epsilon=0,1,\quad \tilde{q}-\epsilon\tilde{p}\in\mathbb{Z}+\mu$	$\alpha_-/\alpha_+=\pm p/q\,$; $c=c^\pm_{pq}$
$N^{23}_\pm$ ($\mathrm{III}^{00}_\pm$)	ϵ	$\epsilon=0,1\quad$ for $\widehat{W}$ $\epsilon=0$ for $\widehat{S},\ \ \epsilon=1$ for $\widehat{R}$,	$c=c^-_{11}=1$ for N^{23}_- $c=c^+_{11}=c_0$ for N^{23}_+